The p-center problem under Uncertainty

Homa Ataei Kachooei* Mansoor Davoodi† Dena Tayebi ‡

Abstract

The problem of p-center asks finding the location of p facilities among a set of n demand points such that the maximum distance between any demand point and its nearest facility is minimized. In this paper, we study the p-center problem under uncertainty, that is, the demand set is given as a set of regions, e.g., n disks. We focus on *Max-p-center* and *Min-p-center* problems as the natural extensions of this problem in uncertainty context. In these problems, we are interested in computing the upper and lower bounds on the p-center solution for the regions. Precisely, we wish to place a point in each region such that the solution of p-center problem for that *placement* is maximized or minimized. We present a $\frac{1}{2}$-approximation and a parameterized approximation algorithm for the Max-p-center and a parameterized approximation algorithm for the Min-p-center problem.

keywords: Facility location, p-center, Uncertainty, Min-p-center, Max-p-center.

1 Introduction

The p-center problem is a classical facility location problem; given n demand points (customers), and the goal is to place p facilities (*centers*) among them such that the maximum distance between any demand point and its nearest center is minimized. It was proved that the p-center problem is NP-hard for both the Euclidean and Manhattan metrics [10]. Some special cases of the p-center problem are solvable in polynomial time such as the smallest enclosing circle and its weighted demand set variations [3, 7, 9], the two-center problem [1], the rectilinear three-center problem [5], the p-center problem on trees [11, 2] and the p-center problem in one dimension [12]. The p-center problem has been also studied under uncertainty; the location of the demand points and the weight of demands may be considered as the uncertainty sources. Further, in the graph variation of the problem, the location and the weight of vertices and the length of edges can be considered uncertain. There are different approaches for modeling uncertainty.

Uncertainty can be modeled by continuous or discrete sets. In continuous model, uncertainty is modeled by some regions or intervals, however, in the discrete model, it is modeled by some discrete sets. Foul [4] studied the Euclidean 1-center problem under uncertainty in which each demand has a uniform distribution in a given rectangle in the plane. The p-center problem was studied in one dimension such that the location of each demand is uncertain [13]. Uncertainty is modeled using m possible locations with a probability distributed function for each demand point. For this problem an $O(mn \log mn + n \log p \log n)$ time algorithm was presented. Also, the 1-center problem in one dimension, the 1-center problem on a tree and the rectilinear 1-center problem in the plane were studied under this model of uncertainty [14, 16, 15]. Löffler and van Kreveld [8] presented efficient algorithms for 1-center problem when the uncertainty regions are modeled by squares or disks. The goal is finding a point from each region such that the *Smallest Enclosing Circle* (SEC) of them is minimized or maximized.

The problems of p-center under uncertainty can be formally defined as follows. Let $D = \{d_1, d_2, ..., d_n\}$ be a set of n disks in the plane and $I = \{p_1, p_2, ..., p_n\}$ be a *placement*, where $p_i \in d_i$ for $i = 1, 2, ..., n$. Now, I is an input or *instance* of the (certain) p-center problem. Let $p - center(I)$ be the optimal solution of the p-center problem for I. That is, if $C = \{c_1, c_2, ..., c_p\}$ is a solution (set of p centers) for p-center problem, then

$$p - center(I) = \min_{C} \max_{p_i \in I} dis(p_i, C),$$

where $dis(p_i, C)$ is the distance between p_i and the nearest center in C. Therefor, *Max-p-center* and *Min-p-center* are the problems of finding the crucial instances I^{max} and I_{min} such that

$$I^{max} : \max_{I} p - center(I).$$

$$I^{min} : \min_{I} p - center(I).$$

In this paper, we consider both Max-p-center and Min-p-center problems. In section 2, we present a simple $\frac{1}{2}$-approximation algorithm for the Max-p-center problem when the regions are disjoint disks or a set of discrete points. Also, we present a $1 - \frac{2}{k+4}$-approximation algorithm when the regions are *k-separable*. In section 3, we consider the Min-p-center problem and present a $1 + \frac{2}{k}$-approximation algorithm when the regions of uncertainty are k-separable disks or discrete points.

*ataei.homa@gmail.com
†Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan, Iran, mdmonfared@iasbs.ac.ir
‡denatayebi@yahoo.com

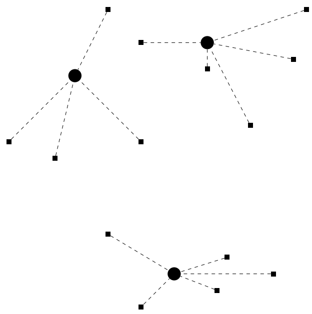


Figure 1: Clustering for three facilities

2 Max-p-center Problem

In this section, we focus on the *Max-p-center* problem when the regions of uncertainty are modeled as disjoint disks or discrete sets. We present a $\frac{1}{2}$-approximation algorithm and a parameterized approximation algorithm for the special case of the Max-p-center problem when the regions are *well-separable*.

Through this paper, we point out an assignment of the demand points to the facility centers as a *clustering*. An example of a clustering with three clusters is shown in Figure 1. Two clustering have the same *topology* if the assignments of demand points to the centers are the same.

Theorem 1 *Let D be a set of disjoint disks as the input of the Max-p-center problem. The algorithm that places the center of disks, has $\frac{1}{2}-$approximation ratio.*

Proof. We consider three clusters C_{opt}, C_{c-opt} and C'. C_{opt} is the solution of Max-p-center problem, e.g., I^{max}, C_{c-opt} is the solution of (certain) p-center problem for the center of disks, and C' is the cluster which have the same topology with C_{c-opt} and the same location with I^{max}. We compare C_{c-opt} and C_{opt} using C'. Since, in the p-center problem the goal is minimizing the maximum length edge in the cluster. Let $e_{max-opt}$ be the maximum distance between any demand point and its assigned center in C_{opt}. Actually $e_{max-opt}$ is the greatest edge in the cluster C_{opt}. Similarly, let e_{c-max} and e'_{max} be the greatest edge in C_{c-opt} and C', respectively. Figure 2 illustrates clusters C_{c-opt}, C_{opt} and C'. As the location of demand points in C' and C_{opt} are the same, thus

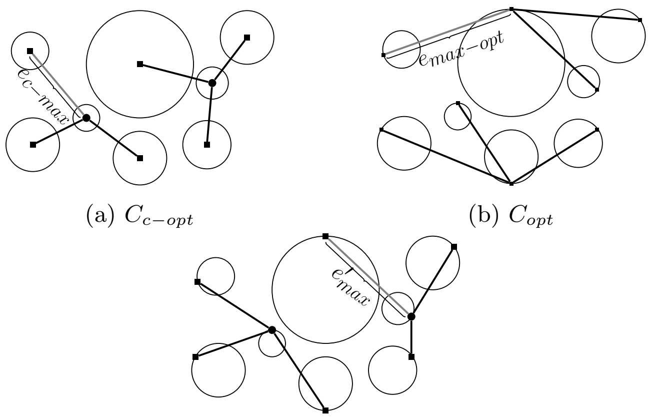
$$e_{max-opt} \leq e'_{max}. \tag{1}$$

Since C_{c-opt} and C' have the same topology, if the location of the demand points changes anywhere on disks, the length of each edge, increases at most the amount of sum of the radius of two (disjoint) disks. So,

$$e'_{max} \leq 2e_{c-max}. \tag{2}$$

According to inequalities 1 and 2, we have

$$e_{max-opt} \leq 2e_{c-max}. \tag{3}$$



(a) C_{c-opt} (b) C_{opt}

(c) C'

Figure 2: Three kinds of clusters for the Max-p-center problem

We compare the corresponding edges in C_{c-opt} and C'. Note that, the largest edges in these two clusters may not be the same in their clustering. It means, the largest edge in C' is between disks D_i and D_j, however, it is between two other different disks in C_{c-opt}. We claim that inequality 2 is established even for this case. Suppose that in C_{c-opt}, e is corresponding edge with e'_{max} in C'. So,

$$e'_{max} \leq 2e. \tag{4}$$

e_{c-max} is the largest edge and e is an edge in C_{c-opt}. Thus,

$$e \leq e_{c-max}. \tag{5}$$

According to inequalitys 4 and 5

$$e'_{max} \leq 2e_{c-max}. \tag{6}$$

Therefor, the inequality 3 is established even for this case. Consequently, the proof is complete. □

Theorem 1 states that the set of center of disks is $\frac{1}{2}-$approximation for I^{max} when the disks are disjoint. In the following, we show that there is a nice relationship between the approximation ratio of such a solution and *separability factor* of the disks by proposing a parametrized approximation ratio.

Let r_{max} be the radius of the largest disk. A set of disks D is called *k-separable*, if the minimum distance between any pair of disks in D is at least $k.r_{max}$. For an input such as D, *separability* is the maximum k such that D is k-separable.

Theorem 2 *Let D be a set of k-separable disks as the input of the Max-p-center problem. The algorithm that places the center of disks, has $1 - \frac{2}{k+4}-$ approximation ratio.*

Proof. This proof is similar to the proof of Theorem 1. We consider C_{c-opt}, C' and C_{opt} as before. Suppose e' is an arbitrary edge in C', and d_i and d_j are two disks connecting with e'. Let r_i and r_j be the radius of d_i and d_j, respectively, and l be the distance between d_i and d_j. Also, let e be the corresponding edge with e' in C_{c-opt} whose weight is $l+r_i+r_j$. The weight of e' is at most $l+2r_i+2r_j$. So, the weight of an edge in C_{c-opt} to the weight of its corresponding edge in C' is at least:

$$\frac{e}{e'} = \frac{l+r_i+r_j}{l+2r_i+2r_j} \geq \frac{k.r_{max}+r_i+r_j}{k.r_{max}+2r_i+2r_j}$$
$$\geq \frac{k.r_{max}+r_{max}+r_{max}}{k.2r_{max}+2r_{max}+r_{max}} = \frac{k+2}{k+4}. \quad (7)$$

This inequality holds for any edge in C_{c-opt}. So, regarding the inequality 7,

$$e_{c-max} \geq \frac{k+2}{k+4} e'_{max}, \quad (8)$$

where e_{c-max} is the edge with maximum weight in C_{c-opt} and e'_{max} is the edge with maximum weight in C'. C_{opt} and C' have the same demand points, so,

$$e_{max-opt} \leq e'_{max}, \quad (9)$$

where $e_{max-opt}$ is the edge with maximum weight in C_{opt}. According to inequalities 8 and 9:

$$e_{c-max} \geq \frac{k+2}{k+4} e_{max-opt}. \quad (10)$$

Therefor, the set of center of the disks is $\frac{k+2}{k+4} = 1 - \frac{2}{k+4}-$ approximation solution. □

Finally, we show that the idea behind of the parametrized approximation algorithm can be applied for the uncertain demand regions modeled by discrete sets with preserving the approximation ratio . We are given a set of points for each uncertainty region, e.g., $S = \{S_1, S_2, ..., S_n\}$, e.g., $I = \{s_1, s_2, ..., s_n\}$, where $s_i \in S_i$ is an *instance* of S_i, for $i = 1, 2, ..., n$, and the goal is choosing a point from each set, such that $p - center(I)$ is maximized. Similarly, it is called, S is *k-separable* if the minimum distances between any pair of uncertainty regions are not less than k times of the maximum distance between different instances of any uncertain region.

Theorem 3 *Let $S = \{S_1, S_2, ..., S_n\}$ be a set of k-separable uncertainty regions modeled by set of discrete points. The problem of Max-p-center can be solved with $\frac{k+2}{k+4} = 1 - \frac{2}{k+4}$ approximation ratio.*

Proof. The proof is similar to the proof of Theorem 2. It is sufficient to choose the solution of the 1-center problem for each set S_i, $1 \leq i \leq n$, instead of placing the center of the disks as the solution. Since the solution of 1-center for each set S_i is a point of S_i whose maximum distance from the other points of S_i is minimized, it satisfies the necessary conditions for the proof. □

2.1 Min-p-center Problem

As explained, the goal of Min-p-center problem is finding an instance I^{min} that minimizes $p-center(I)$ among the all possible instances I of the uncertainty regions. We show again that the idea of placing center of the uncertainty regions resulted in good approximation of I^{min}, e.g., a $1 + \frac{2}{k}-$approximation solution when the regions are k-separable.

Theorem 4 *Let D be a set of k-separable disks as the input of the Min-p-center problem. The algorithm that places the center of disks is a $1 + \frac{2}{k}-$ approximation algorithm.*

Proof. This proof is similar to the proof of Theorem 2, however, the definition of the clusters is different. Let C_{opt} be the solution of Min-p-center problem, C_{c-opt} be the solution of p-center for the center of the disks and C' be the cluster which has the same topology with C_{opt} and the same location of demand points with C_{c-opt}. Since both C_{c-opt} and C' are the clusters on the center of disks and C_{c-opt} is the optimal solution of the p-center problem, we have

$$e_{c-max} \leq e'_{max} \quad (11)$$

where e_{c-max} is the edge with maximum weight in C_{c-opt} and e'_{max} is the edge with maximum weight in C'.

We consider an arbitrary edge $e' \in C'$. Suppose d_i and d_j are two connecting disks by e'. Let r_i and r_j be the radius of d_i and d_j, respectively, and l be the maximum distance between d_i and d_j. In C_{opt}, d_i and d_j connect to each other by an edge e which its weight is at least l. The weight of e' is at most $l + r_i + r_j$. So, the weight of an edge in C_{opt} to the weight of its corresponding edge in C' is at least

$$\frac{e}{e'} = \frac{l}{l+r_i+r_j} \geq \frac{k.r_{max}}{k.r_{max}+r_i+r_j}$$
$$\geq \frac{k.r_{max}}{k.r_{max}+r_{max}+r_{max}} = \frac{k}{k+2} \quad (12)$$

This is established for any edge in C_{opt} and its corresponding edge in C'. So,

$$e_{max-opt} \geq \frac{k}{k+2} e'_{max}, \quad (13)$$

where $e_{max-opt}$ is the edge in C_{opt} with maximum length. According to inequalities 11 and 13:

$$e_{max-opt} \geq \frac{k}{k+2} e_{c-max}. \quad (14)$$

Thus, the proof is complete. □

Theorem 5 *The problem Min-p-center for a set of $k-$separable discrete sets can be solved with the $\frac{k+2}{k} = 1 + \frac{2}{k}$ approximation ratio.*

Proof. Similar to the proof of Theorem 3 and Theorem 4, it is sufficient to choose the solution of discrete 1-center problem as the instance of Min-p-center problem, and follows the proof of Theorem 4. $\square$

3 Conclusion and Future Work

In this paper, we defined two variations of the problem of p-center in the uncertainty context, the *Max-p-center* problem and the *Min-p-center* problem. In fact, these problems are the natural extension of p-center under uncertainty. In these problems, a set of regions, called *uncertainty regions*, are as given and the goal is placing a point in each region such that the worst and the best case happen for p-center problem, i.e., the instances resulted in maximizing or minimizing the objective value of the p-center problem. We considered two cases for the uncertainty regions, disjoint disks and discrete set of points. We presented a $\frac{1}{2}$-approximation algorithm and a parameterized approximation algorithm for the Max-p-center problem and a parameterized approximation algorithm for the Min-p-center problem.

In addition to the extension of the p-center problem under uncertainty defined in this paper, there is another extension called *Max-Regret* [6]. The *regret* is defined as the difference between the cost of a given solution and the cost of the optimal solution for a particular placement of the uncertain points. The worst case of *regret* between all possible placement of the uncertain points is called Max-Regret. So, a potential direction for future work include consideration of *Max-Regret p-center problem*.

References

[1] T. M. Chan. More planar two-center algorithms. *Computational Geometry*, 13(3):189–198, 1999.

[2] R. Chandrasekaran and A. Tamir. Polynomially bounded algorithms for locatingp-centers on a tree. *Mathematical Programming*, 22(1):304–315, 1982.

[3] M. E. Dyer. On a multidimensional search technique and its application to the euclidean one-centre problem. *SIAM Journal on Computing*, 15(3):725–738, 1986.

[4] A. Foul. A 1-center problem on the plane with uniformly distributed demand points. *Operations Research Letters*, 34(3):264–268, 2006.

[5] M. Hoffmann. A simple linear algorithm for computing rectilinear 3-centers. *Computational Geometry*, 31(3):150–165, 2005.

[6] P. Kouvelis, G. L. Vairaktarakis, and G. Yu. *Robust 1-median location on a tree in the presence of demand and transportation cost uncertainty.* Department of Industrial & Systems Engineering, University of Florida, 1993.

[7] D. Lee and Y.-F. Wu. Geometric complexity of some location problems. *Algorithmica*, 1(1-4):193, 1986.

[8] M. Löffler and M. van Kreveld. Largest bounding box, smallest diameter, and related problems on imprecise points. *Computational Geometry*, 43(4):419–433, 2010.

[9] N. Megiddo. Linear-time algorithms for linear programming in rˆ3 and related problems. *SIAM journal on computing*, 12(4):759–776, 1983.

[10] N. Megiddo and K. J. Supowit. On the complexity of some common geometric location problems. *SIAM journal on computing*, 13(1):182–196, 1984.

[11] N. Megiddo and A. Tamir. New results on the complexity of p-centre problems. *SIAM Journal on Computing*, 12(4):751–758, 1983.

[12] N. Megiddo, A. Tamir, E. Zemel, and R. Chandrasekaran. An $o(n \log^2 n)$ algorithm for the k th longest path in a tree with applications to location problems. *SIAM Journal on Computing*, 10(2):328–337, 1981.

[13] H. Wang and J. Zhang. One-dimensional k-center on uncertain data. *Theoretical Computer Science*, 602:114–124, 2015.

[14] H. Wang and J. Zhang. A note on computing the center of uncertain data on the real line. *Operations Research Letters*, 44(3):370–373, 2016.

[15] H. Wang and J. Zhang. Computing the center of uncertain points on tree networks. *Algorithmica*, 78(1):232–254, 2017.

[16] H. Wang and J. Zhang. Computing the rectilinear center of uncertain points in the plane. *International Journal of Computational Geometry and Applications*, 28(03):271–288, 2018.